Physical Hamiltonian on the Physical Hilbert Space for a Model in NR QED

Joint work with F. Hiroshima (Kyushu Univ.)

Setsunan Univ., Feb. 24, 2009

Akito Suzuki (Kyushu Univ.)

e-mail: sakito@math.kyushu-u.ac.jp

0. Introduction

Charged particle in the quantized electromagnetic field:

Hamiltonian in the dipole approximation is

$$H(\boldsymbol{p}) = \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A}(\varphi))^2 + H_{\mathsf{f}} + eA_0(\varphi),$$

where

$m > 0$ and $e \in \mathbb{R}$: mass and charge of the particle
$\boldsymbol{p} \in \mathbb{R}^3$: momenta of the particle
$\varphi : \mathbb{R}^3 \to \mathbb{R}$: charge distribution
H_{f} : Hamiltonian of the electromagnetic field
$(\boldsymbol{A}, A_0)$: electromagnetic vector potential and

$$A_\mu(\varphi) := \int A_\mu(x)\varphi(x).$$

Decomposition of the vector field:

$$\boldsymbol{A} = \boldsymbol{A}^{\perp} + \boldsymbol{A}^{\parallel}, \quad A_0 = A^{\mathsf{s}}$$

$$H_{\mathsf{f}} = H_{\mathsf{f}}^{\perp} + H_{\mathsf{f}}^{\parallel} + H_{\mathsf{f}}^{\mathsf{s}},$$

where $^{\perp}$, $^{\parallel}$ and $^{\mathsf{s}}$ denotes transversal, longitudinal and scalar:

$$\boldsymbol{\nabla} \cdot \boldsymbol{A}^{\perp} = 0.$$

Coulomb gauge: $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$, i.e., $\boldsymbol{A}^{\parallel} = 0$.

$$H_{\mathsf{Coul}}(\boldsymbol{p}) = \frac{1}{2m}\left(\boldsymbol{p} - e\boldsymbol{A}^{\perp}(\varphi)\right)^2 + H_{\mathsf{f}}^{\perp} + \frac{e^2}{2}E_{\varphi},$$

where $E_{\varphi} = \|\widehat{\varphi}/\omega\|^2_{L^2(\mathbb{R}^3)}$ with $\omega(k) = |k|$.

Quantization : $[A_i^{\perp}(x), \dot{A}_j^{\perp}(y)] = i\delta_{ij}^{\perp}(x - y)$.

Known result(A. Arai '81,'83) Suppose $\left\|\frac{\hat{\varphi}}{\omega^{1/2}}\right\|_{L^2(\mathbb{R}^3)}, \left\|\frac{\hat{\varphi}}{\omega}\right\|_{L^2(\mathbb{R}^3)} < \infty$. Then under some conditions for φ:

- $H_{\mathsf{Coul}}(\boldsymbol{p})$ is defined on the Hilbert space

$$\Gamma(\oplus^2 L^2(\mathbb{R}^3)) = \bigoplus_{n=0}^{\infty} \otimes_{\mathsf{s}}^n \left[\oplus^2 L^2(\mathbb{R}^3)\right]$$

  and

$$H_{\mathsf{Coul}}(\boldsymbol{p}) \text{ : self-adjoint on } D(H_{\mathsf{f}}^{\perp})$$

- $\sigma(H_{\mathsf{Coul}}(\boldsymbol{p})) \subset \mathbb{R}$ and $E(H_{\mathsf{Coul}}(\boldsymbol{p})) := \inf \sigma(H_{\mathsf{Coul}}(\boldsymbol{p})) > -\infty$.

- $\left\|\frac{\hat{\varphi}}{\omega^{3/2}}\right\|_{L^2(\mathbb{R}^3)} < \infty \Rightarrow H_{\mathsf{Coul}}(\boldsymbol{p})$ has a unique ground state.

- $\left\|\frac{\hat{\varphi}}{\omega^{3/2}}\right\|_{L^2(\mathbb{R}^3)} = \infty$ and $\boldsymbol{p} \neq 0 \Rightarrow H_{\mathsf{Coul}}(\boldsymbol{p})$ has no ground state.

Contents:

1. Definitions

Quantization:

$$[A_\mu(x), \dot{A}_\nu(y)] = -ig_{\mu\nu}\delta(x-y)$$

with

$$g_{00} = 1 = -g_{jj} \quad (j = 1, 2, 3), \quad g_{\mu\nu} = 0 \quad (\mu \neq \nu).$$

Let us define

$$H(\boldsymbol{p}) = \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A}(\varphi))^2 + H_{\mathsf{f}} + eA_0(\varphi)$$

on

$$\mathcal{H} := \Gamma(\oplus^4 L^2(\mathbb{R}^3)) = \bigoplus_{n=0}^{\infty} \otimes_{\mathsf{s}}^n \left[\oplus^4 L^2(\mathbb{R}^3)\right].$$

Creation and annihilation ops: $a^*(f;\mu)$ and $a(g;\nu)$ satisfy

- $a^*(f;\mu) = a(\bar{f};\mu)^*, \quad f \in L^2(\mathbb{R}^3), \quad \mu = 1,2,3,0;$
- $[a(f;\mu), a^*(g;\nu)] = \delta_{\mu\nu}(\bar{f}, g)_{L^2(\mathbb{R}^3)}, \quad f, g \in L^2(\mathbb{R}^3), \quad \mu,\nu = 1,2,3,0$
- for the vacuum $\Omega := \{1,0,0,\cdots\} \in \Gamma(\oplus^4 L^2(\mathbb{R}^3))$,

$$a(f;\mu)\Omega = 0, \quad f \in L^2(\mathbb{R}^3), \quad \mu = 1,2,3,0.$$

Denote

$$a(f;\mu) = \int f(k)a(k;\mu)dk, \quad a^*(f;\mu) = \int f(k)a^*(k;\mu)dk.$$

Hamiltonian of the field:

$$H_{\mathsf{f}} := \int \omega(k)\left[\sum_{\mu=0}^{3} a^*(k;\mu)a(k;\mu)\right] dk$$

with $\omega(k) = |k|$.

Field operators:

$$\boldsymbol{A}(x) = \int \frac{dk}{\sqrt{2(2\pi)^3\omega(k)}} \sum_{l=1}^{3} \left(e^{ik\cdot x}a(k;l) + e^{-ikx}a^*(k;l)\right) \boldsymbol{e}^{(l)}(k)$$

$$A_0(x) = \int \frac{dk}{\sqrt{2(2\pi)^3\omega(k)}} \left(e^{ik\cdot x}a(k;0) - e^{-ikx}a^*(k;0)\right),$$

where $\boldsymbol{e}^{(l)}(k) \in \mathbb{R}^3$ is the polarization vector:

$$\boldsymbol{e}^{(l)}(k) \cdot \boldsymbol{e}^{(l')}(k) = \delta_{ll'}, \quad \boldsymbol{e}^{(3)} := \frac{k}{|k|}.$$

$$\dot{\boldsymbol{A}}(x) = -i\int dk \sqrt{\frac{\omega(k)}{2(2\pi)^3}} \sum_{l=1}^{3} \left(e^{ik\cdot x}a(k;l) - e^{-ikx}a^*(k;l)\right) \boldsymbol{e}^{(l)}(k)$$

$$\dot{A}_0(x) = -i\int dk \sqrt{\frac{\omega(k)}{2(2\pi)^3}} \left(e^{ik\cdot x}a(k;0) + e^{-ikx}a^*(k;0)\right),$$

Remark:

- H_{f} and $\boldsymbol{A}$ are self-adjoint but A_0 not even Hermitian.

- $\boldsymbol{A} = \boldsymbol{A}^{\perp} + \boldsymbol{A}^{\|}$:

$$\boldsymbol{A}^{\perp}(x) = \int \frac{dk}{\sqrt{2(2\pi)^3\omega(k)}} \sum_{l=1}^{2} \left(e^{ik\cdot x} a(k;l) + e^{-ikx} a^*(k;l) \right) \boldsymbol{e}^{(l)}(k)$$

$$\boldsymbol{A}^{\|}(x) = \int \frac{dk}{\sqrt{2(2\pi)^3\omega(k)}} \left(e^{ik\cdot x} a(k;3) + e^{-ikx} a^*(k;3) \right) \boldsymbol{e}^{(3)}(k)$$

- $H_{\mathsf{f}} = H_{\mathsf{f}}^{\perp} + H_{\mathsf{f}}^{\|} + H_{\mathsf{f}}^{\mathsf{s}}$:

$$H_{\mathsf{f}} = \int \omega(k) \left[\sum_{l=1}^{2} a^*(k;l) a(k;l) + a^*(k;3) a(k;3) + a^*(k;0) a(k;0) \right] dk$$

Let

$$\eta\ :\ \text{a unitary self-adjoint operator}$$

such that $\eta = \eta^* = \eta^{-1}$, $\eta\Omega = \Omega$ and

$$\eta a(f;\mu)\eta = (-1)^{\delta_{\mu 0}} a(f;\mu), \quad \eta a^*(f;\mu)\eta = (-1)^{\delta_{\mu 0}} a^*(f;\mu).$$

PROPOSITION Suppose $\widehat{\varphi}/\omega^{1/2}$, $\widehat{\varphi}/\omega^{-1} \in L^2(\mathbb{R}^3)$. Then:

$$H(\boldsymbol{p}) = \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A}(\varphi))^2 + H_{\mathsf{f}} + eA_0(\varphi)$$

is a closed operator with the domain $D(H(\boldsymbol{p})) = D(H_{\mathsf{f}})$ and

$$H(\boldsymbol{p}) \text{ is } \eta\text{-self-adjoint}$$

i.e.,

$$H(\boldsymbol{p})^\dagger := \eta H(\boldsymbol{p})^* \eta = H(\boldsymbol{p}).$$

New metric: Let

$$\langle \Psi \mid \Phi \rangle := (\Psi, \eta \Phi)_{\mathcal{H}}, \qquad \Psi, \Phi \in \mathcal{H}.$$

Then:

- For $\Psi \in D(H(\boldsymbol{p}))$,

$$\langle \Psi \mid H(\boldsymbol{p}) \Psi \rangle = \langle H(\boldsymbol{p}) \Psi \mid \Psi \rangle \in \mathbb{R}.$$

- Let $a^\dagger(f; \mu) = a^*(f; \mu)$. Then $\langle \Omega \mid \Omega \rangle = 1 > 0$ and

$$\langle a^\dagger(f; 0)\Omega \mid a^\dagger(f; 0)\Omega \rangle = -\|f\|_{L^2(\mathbb{R}^3)} < 0.$$

$$\therefore \langle \cdot \mid \cdot \rangle \text{ is an indefinite metric!}$$

2. Results

Assumptions:

- $\exists\epsilon > 0$ such that $\|e^{+\epsilon\omega}\widehat{\varphi}\|_\infty < \infty$,
- $\exists\rho$ on $[0,\infty)$ such that $\widehat{\varphi}(k) = \rho(|k|)$,
- $\rho(s) > 0$ for $s \neq 0$,
- $F(s) := \rho(\sqrt{s})^2\sqrt{s} \in L^p([0,\infty); ds)$ for some $1 < p$,
- $0 < \exists\alpha < 1$ s.t.

  $$|F(s+h) - F(s)| \leq K|h|^\alpha$$

  for all s and $0 < h \leq 1$.

PROPOSITION (Existence of dynamics and asymptotic field)

- There exists a solution $A_\mu(x,t)$ of the Heisenberg equation

$$\frac{d}{dt}A_\mu(x,t) = i[H(\boldsymbol{p}), A_\mu(x,t)]$$

with $A_\mu(x,0) = A_\mu(x)$.

- There exist the limits

$$A_\mu^{\pm}(x,t)\Psi := \lim_{s\to\pm\infty} A_\mu^s(x,t)\Psi,$$

where $A_\mu^s(x,t)$ $(s \in \mathbb{R})$ is the free field such that

$$\frac{d^2}{dt^2}A_\mu^s(x,t)\Psi - \Delta A_\mu^s(x,t)\Psi = 0$$

with $A_\mu^s(x,s) = A_\mu(x,s)$ and $\dot{A}_\mu^s(x,s) = \dot{A}_\mu(x,s)$.

Remark: Formally,

- $A_\mu(x,t)$ is given by

$$A_\mu(x,t) = e^{itH(\boldsymbol{p})} A_\mu(x) e^{-itH(\boldsymbol{p})}$$

  but $e^{-itH(\boldsymbol{p})}$is ??

- $A^s_\mu(x,t)$ is given by

$$A^s_\mu(x,t) = e^{isH(\boldsymbol{p})} e^{-i(t-s)H_{\mathsf{f}}} A_\mu(x) e^{i(t-s)H_{\mathsf{f}}} e^{-isH(\boldsymbol{p})}.$$

THEOREM (Physical subspace)

Suppose $\left\|\frac{\hat{\varphi}}{\omega^{3/2}}\right\| < \infty$. Then

$$\exists \mathcal{V}^{\pm}_{\text{phys}} \subset \Gamma(\oplus^4 L^2(\mathbb{R}^3))$$

such that:

- $\mathcal{V}^{\pm}_{\text{phys}}$ is non-trivial, closed and positive semi-definite, i.e.

$$\mathcal{V}^{\pm}_{\text{phys}} \neq \{0\}, \quad \langle \Psi \mid \Psi \rangle \geq 0, \quad \Psi \in \mathcal{V}^{\pm}_{\text{phys}}.$$

- For $\Psi \in \mathcal{V}^{\pm}_{\text{phys}}$,

$$\left\langle \Psi \middle| \left[\frac{d}{dt} A^{\pm}_0(x,t) + \boldsymbol{\nabla} \cdot \boldsymbol{A}^{\pm}(x.t) \right] \Psi \right\rangle = 0.$$

- $\mathcal{V}^{+}_{\text{phys}} \neq \mathcal{V}^{-}_{\text{phys}}$.

LEMMA

- $H(\boldsymbol{p})\left(\mathcal{V}^{\pm}_{\text{phys}} \cap D(H(\boldsymbol{p}))\right) \subset \mathcal{V}^{\pm}_{\text{phys}}$.

- $K^{\pm}(\boldsymbol{p}) := H(\boldsymbol{p})\Big|_{\mathcal{V}^{\pm}_{\text{phys}}}$ is a densely defined closed operator and

$$K^{\pm}(\boldsymbol{p})\mathcal{V}^{\pm}_{\text{null}} \subset \mathcal{V}^{\pm}_{\text{null}},$$

where

$$\mathcal{V}^{\pm}_{\text{null}} := \{\Psi \in \mathcal{V}^{\pm}_{\text{phys}} \mid \langle \Psi \mid \Psi \rangle = 0\}.$$

DEFINITION

Physical Hilbert space:

$$\mathcal{H}^{\pm}_{\text{phys}} := \mathcal{V}^{\pm}_{\text{phys}}/\mathcal{V}^{\pm}_{\text{null}}$$

with

$$\langle [\Psi_\pm]_\pm, [\Phi_\pm]_\pm \rangle_{\mathcal{H}^{\pm}_{\text{phys}}} := \langle \Psi_\pm \mid \Phi_\pm \rangle, \qquad \Psi_\pm, \Phi_\pm \in \mathcal{V}^{\pm}_{\text{phys}}.$$

Physical Hamiltonian:

$$H^{\pm}_{\text{phys}}(\boldsymbol{p}) := z + \left[\left(K^{\pm}(\boldsymbol{p}) - z \right)^{-1} \right]_{\pm}^{-1} \quad \text{with some } z \in \mathbb{C} \setminus \text{spec}(H).$$

Here, for a bounded operator L on $\mathcal{V}^{\pm}_{\text{phys}}$,

$[L]_\pm [\Psi]_\pm = [L\Psi]_\pm,$

$[\Psi]_\pm$: the element of $\mathcal{H}^{\pm}_{\text{phys}}$ associated with $\Psi \in \mathcal{V}^{\pm}_{\text{phys}}$

is well-defined if $L\mathcal{V}^{\pm}_{\text{null}} \subset \mathcal{V}^{\pm}_{\text{null}}$.

THEOREM (Physical Hilbert space and Hamiltonian)
Suppose $\widehat{\varphi}/\omega^{3/2} \in L^2(\mathbb{R}^3)$. Then:

- $H^{\pm}_{\text{phys}}(\boldsymbol{p})$ is well-defined, self-adjoint and

$$H^{\pm}_{\text{phys}}(\boldsymbol{p}) \sim \frac{\boldsymbol{p}^2}{2m_{\text{eff}}} + H_{\text{f}}^{\perp} + E_0 + E_1 \quad \text{on } \Gamma(\oplus^2 L^2(\mathbb{R}^3)),$$

where

$$H_{\text{f}}^{\perp} = \int \omega(k) \sum_{l=1,2} a^*(k;l)a(k;l)dk$$

$$m_{\text{eff}} = m + 2E_0, \quad E_0 = \frac{e^2}{2}\|\widehat{\varphi}/\omega\|^2_{L^2(\mathbb{R}^3;dk)},$$

$$E_1 = \frac{3}{2\pi}\int_{-\infty}^{\infty} \frac{e^2 s^2 \|\widehat{\varphi}/(s^2+\omega^2)\|^2_{L^2(\mathbb{R}^3;dk)}}{m + e^2\|\widehat{\varphi}/\sqrt{s^2+\omega^2}\|^2_{L^2(\mathbb{R}^3;dk)}} ds.$$

- $H^{\pm}_{\text{phys}}(\boldsymbol{p})$ has a unique ground state.

3. Proofs

Fix $\boldsymbol{p} \in \mathbb{R}^3$.

- Define $A_0(x,t)$ by the solution of

$$\Box A_0(x,t) = e\varphi(x),$$

which is exactly solvable. Then

$$\frac{d}{dt} A_0(x,t) = i[H_{\mathrm{f}}^{\mathrm{s}} + A_0(\varphi), A_0(x,t)] = i[H(\boldsymbol{p}), A_0(x,t)].$$

- Let $H^{\perp+\parallel}(\boldsymbol{p}) = H(\boldsymbol{p}) - H_{\mathrm{f}}^{\mathrm{s}} - A_0(\varphi)$ and define

$$\boldsymbol{A}(x,t) = e^{itH^{\perp+\parallel}(\boldsymbol{p})} \boldsymbol{A}(x) e^{-itH^{\perp+\parallel}(\boldsymbol{p})}, \quad j = 1,2,3.$$

Then

$$\frac{d}{dt} A_j(x,t) = i[H^{\perp+\parallel}(p), A_j(x,t)] = i[H(\boldsymbol{p}), A_j(x,t)].$$

Exact solution of the Heisenberg equation:

THEOREM Let $\left\|\frac{\hat{\varphi}}{\omega^{3/2}}\right\| < \infty$. Then

$$\exists U(\boldsymbol{p}) \text{ (unitary)}$$

such that

$$b(f,j) = U(\boldsymbol{p})a(f,j)U(\boldsymbol{p})^{-1}, \quad f \in L^2(\mathbb{R}^3)$$

satisfy the following:

$$\begin{aligned}
&[b(f,i), b^\dagger \dagger (g,j)] = \delta_{ij}(\bar{f}, g)_{L^2(\mathbb{R}^3)},\\
&[b(f,i), b(g,j)] = [b^\dagger(f,i), b^\dagger(g,j)] = 0,\\
&[H(\boldsymbol{p}), b^\dagger(f,j)] = b^\dagger(\omega f, j)\\
&[H(\boldsymbol{p}), b(f,j)] = -b(\omega f, j).
\end{aligned}$$

Moreover, $\Omega(\boldsymbol{p}) = U(\boldsymbol{p})\Omega$ is a unique vector satisfying

$$b(f,j)\Omega(\boldsymbol{p}) = 0, \quad f \in L^2(\mathbb{R}^3),\ j = 1,2,3$$

up to constant factor.

Proof. Let

$$Tf = f + e^2 Q\sqrt{\omega}G\sqrt{\omega}\widehat{\varphi}f,$$

where $G = \lim_{\epsilon\downarrow 0} G_\epsilon$,

$$G_\epsilon f(k) = \int_{\mathbb{R}^3} dk' \frac{f(k')}{(\omega(k)^2 - \omega(k')^2 + i\epsilon)\sqrt{\omega(k)\omega(k')}}$$

$$Q(k) = \frac{\widehat{\varphi}(k)}{D_+(\omega(k)^2)}, \qquad D_\pm(s) = \lim_{\epsilon\downarrow 0} D(s \pm i\epsilon)$$

with

$$D(z) = m - e^2 \int_{\mathbb{R}^3} dk \frac{\widehat{\varphi}(k)^2}{z - \omega(k)^2}.$$

Let

$$b(f,j) = \sum_{i=1}^{3} \left(a^\dagger(W_-^{ij} f, i) + a(W_+^{ij} f, i) - e\frac{p_i}{\sqrt{2}} \left\langle \frac{e_i^{(j)} Q}{\omega^{3/2}}, f \right\rangle \right)$$

$$b^\dagger(f,j) = \sum_{i=1}^{3} \left(a^\dagger(\bar{W}_+^{ij} f, i) + a(\bar{W}_-^{ij} f, i) - e\frac{p_i}{\sqrt{2}} \left\langle \frac{e_i^{(j)} \bar{Q}}{\omega^{3/2}}, f \right\rangle \right),$$

where

$$W_+^{ij} f = \frac{1}{2} \sum_{l=1}^{3} e_l^{(i)} \left(\frac{1}{\sqrt{\omega}} T^* \sqrt{\omega} + \sqrt{\omega} T^* \frac{1}{\sqrt{\omega}} \right) e_l^{(j)} f,$$

$$W_-^{ij} f = \frac{1}{2} \sum_{l=1}^{3} e_l^{(i)} \left(\frac{1}{\sqrt{\omega}} T^* \sqrt{\omega} - \sqrt{\omega} T^* \frac{1}{\sqrt{\omega}} \right) \widetilde{e_l^{(j)}} \tilde{f}.$$

The map

$$\mathbb{W} = \begin{bmatrix} W_+ & \bar{W}_- \\ W_- & \bar{W}_+ \end{bmatrix} : \bigoplus^2 \left[\bigoplus^3 L^2(\mathbb{R}^3) \right] \longrightarrow \bigoplus^2 \left[\bigoplus^3 L^2(\mathbb{R}^3) \right]$$

has the symplectic structure:

$$\mathbb{W}^* J \mathbb{W} = \mathbb{W} J \mathbb{W}^* = J, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and W_- is a Hilbert-Schmidt operator. $\Box$

THEOREM (Exact solution of the Heisenberg equation)

$$\boldsymbol{A}(f,t) = \frac{1}{\sqrt{2}} \sum_{i=1}^{3} \left[b^\dagger \left(\frac{e^{it\omega} \boldsymbol{e}^{(i)}}{\sqrt{\omega}} \bar{T} \widehat{f}, i \right) + b \left(\frac{e^{-it\omega} \boldsymbol{e}^{(i)}}{\sqrt{\omega}} T \widetilde{\widehat{f}}, i \right) \right]$$
$$+ \frac{e\boldsymbol{p}}{m_{\mathsf{eff}}} \left\langle \frac{\widehat{\varphi}}{\omega^{3/2}}, \frac{\widehat{f}}{\sqrt{\omega}} \right\rangle,$$
$$\dot{\boldsymbol{A}}(f,t) = \frac{1}{\sqrt{2}} \sum_{i=1}^{3} \left[b^\dagger \left(e^{it\omega} \sqrt{\omega} \boldsymbol{e}^{(i)} \bar{T} \widehat{f}, i \right) + b \left(e^{-it\omega} \sqrt{\omega} \boldsymbol{e}^{(i)} T \widetilde{\widehat{f}}, i \right) \right].$$

Let

$$a_-(h,j) = b(h,j), \quad a_+^\dagger(h,j) = b^\dagger(h,j),$$
$$a_+(h,j) = \sum_{i=1}^{3} b(L^{ij}h, i), \quad a_+^\dagger(h,j) = \sum_{i=1}^{3} b^\dagger(\bar{L}^{ij}h, i),$$

where

$$L^{ij}h = \delta_{ij}h - i\pi e^2 Q \widehat{\varphi} \omega \sum_{l=1}^{3} e_l^{(i)} [e_l^{(j)} h]$$

with $[f](k) = \int_{S_2} f(|k|\Omega) d\Omega$.

THEOREM (Asymptotic field)

$$\boldsymbol{A}^{\pm}(x,t) = \int \frac{dk}{\sqrt{2(2\pi)^3\omega(k)}} \times \sum_{l=1}^{3} \left(e^{-i\omega(k)t+ik\cdot x} a_{\pm}(k;l) + e^{i\omega(k)t-ikx} a_{\pm}^{*}(k;l)\right) \boldsymbol{e}^{(l)}(k)$$

$$A_0^{\pm}(x,t) = \int \frac{dk}{\sqrt{2(2\pi)^3\omega(k)}} \times \left(e^{-i\omega(k)t+ik\cdot x} a_{\pm}(k;0) - e^{i\omega(k)t-ikx} a_{\pm}^{*}(k;0)\right).$$

Then

$$\frac{d}{dt}A_0^{\pm}(x,t) + \nabla \cdot \boldsymbol{A}^{\pm}(x.t) = \int dk \left(e^{-i\omega(k)t+ik\cdot x} c_{\pm}(k) + e^{i\omega(k)t-ikx} c_{\pm}^{\dagger}(k)\right).$$

Define

$$\mathcal{V}_{\text{phys}}^{\pm} := \{\Psi \mid c_{\pm}(k)\Psi = 0\}.$$

4. Concluding Remark

THEOREM (Absence of physical subspace)

Let $\left\|\frac{\widehat{\varphi}}{\omega^{3/2}}\right\| = \infty$. Then

$$\mathcal{V}_{\mathrm{phys}}^{\pm} = \{0\}.$$