New Renormalization for the Field Operators on Constructive QFT by means of the *Hida Product*

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Abstract

A new definition of product between the elements of Hida distributions defined on the sharp time free field is given. By this procedure the space of Hida distributions has the structure of *ring*. This newly defined product is different to the well known *S*-transform. It is much more complicated, but might have more fruitful structures than *S*-transform.

1 Definition of *Hida product*

Let \( d \in \mathbb{N} \ (d \geq 2) \) be a given space time dimension. By making use of stochastic integral expressions by means of the \( d \)-dimensional (space time) Brownian sheet \( B^d(x) \) and \( d - 1 \)-dimensional (space) Brownian sheet \( B^{d-1}(\bar{x}) \), we define two fundamental random fields \( \phi_N \), the *Nelson’s Euclidean free field*, and \( \phi_0 \), the *sharp time free field*, as follows:

For \( d \geq 2 \),

\[
\phi_N(\cdot) \equiv \int_{\mathbb{R}^d} L_{-\frac{1}{2}}(x - \cdot) \hat{B}^d(x) \, dx, \tag{1.1}
\]

\[
\phi_0(\cdot) \equiv \int_{\mathbb{R}^{d-1}} H_{-\frac{1}{2}}(\bar{x} - \cdot) \hat{B}^{d-1}(\bar{x}) \, d\bar{x}. \tag{1.2}
\]

These definitions of \( \phi_N \) and resp. \( \phi_0 \) seems formal, but they are rigorously defined as \( \mathcal{S}'(\mathbb{R}^d) \) and resp. \( \mathcal{S}'(\mathbb{R}^{d-1}) \) valued random variables through a limiting procedure (cf. [AY1,2]).

Let \( \mu_0 \) and \( \mu_N \) be the probability laws of \( \phi_0 \) and \( \phi_N \) respectively.

For real \( \gamma \), let

\[
H_{-\gamma} \equiv (-\Delta_{d-1} + m^2)^{-\gamma}. \tag{1.3}
\]

For \( r \in \mathbb{N} \), let \( \Lambda_{r,d-1} \in C^\infty_0(\mathbb{R}^{d-1} \to \mathbb{R}_+) \) be a given function such that

\[
0 \leq \Lambda_{r,d-1}(\bar{x}) \leq 1 \ (\bar{x} \in \mathbb{R}^{d-1}), \quad \Lambda_{r,d-1} \equiv 1 \ (|\bar{x}| \leq r), \quad \Lambda_{r,d-1} \equiv 0 \ (|\bar{x}| \geq r + 1),
\]

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for $p \in \mathbb{N}$ define a Hida distribution $\langle : \phi_{0}^{2p} ; \Lambda_{r,d-1} >$ on the space the test functions, $\cap_{q \geq 1} L^{q}(\mu_{0})$ as follows:

$$
\langle : \phi_{0}^{2p} ; \Lambda_{r,d-1} > \\
\equiv \int_{(\mathbb{R}^{d-1})^{\infty}} \left\{ \int_{\mathbb{R}^{d-1}} \Lambda_{r,d-1}(\vec{x}) \prod_{k=1}^{2p} H_{-\frac{1}{m}}(\vec{x} - \vec{x}_{k}) \, d\vec{x} \right\}
\times : \hat{B}^{d-1}(\vec{x}_{1}) \cdots \hat{B}^{d-1}(\vec{x}_{2p}) : \, d\vec{x}_{1} \cdots d\vec{x}_{2p},
$$

(1.4)

here, all the way of using notations follow the rule given by Remark 1.

For $d = 2$ $(d - 1 = 1)$ we know that

$$
\langle : \phi_{0}^{2p} ; \Lambda_{r,1} > \in \cap_{q \geq 1} L^{q}(\mu_{0}).
$$

But our main interest is concentrated on the case where $d = 4$ $(d - 1 = 3)$, and in this case $\langle : \phi_{0}^{2p} ; \Lambda_{r,3} >$ is not a random variable any more, but a Hida distribution.

In the sequel, if there is no indication of the dimension $d$ in each discussion, then we should understand that the consideration is carried out on $d = 4, d - 1 = 3$.

Let us define a new multiplication between two Hida distributions $\langle : \phi_{0}^{2p} ; \Lambda_{r,3} >$ and $\langle : \phi_{0}^{2p} ; \Lambda_{r,3} >$. We name this new multiplication procedure as the Hida product. It produces one Hida distribution from another two, and the resulting distributions are different from the ones derived through the well known S-transform and others. Hence, equipping this multiplication, the space of Hida distribution has the structure of ring.

We have to stress that the Hida distributions generated through this new multiplication procedure are much more complicated than the ones given by the known multiplication procedures, but they are much more fruitful, in fact by these distributions (operators) we may define the non-trivial interactions on the 4-dimensional space time quantum field.

**Definition. (Hida product)** For the space time dimension $d = 4$, let $\phi_{0}$ be the sharp time free field with $d - 1 = 3$. Let the Hida distribution $\langle : \phi_{0}^{2p} ; \Lambda_{r,3} >$ be the $2p$-th Wick power of the sharp time free field defined by (1.4). The **Hida product**

$$
\langle : \phi_{0}^{2p} ; \Lambda_{r,3} > \times \mathcal{H} \langle : \phi_{0}^{2p} ; \Lambda_{r,3} >
$$

$$
= \langle : \phi_{0}^{2p} ; \Lambda_{r,3} > \times \langle : \phi_{0}^{2p} ; \Lambda_{r,3} >
$$

$$
- (\text{all the terms that are not Hida distributions}),
$$

(1.5)

explicitly

$$
\langle : \phi_{0}^{2p} ; \Lambda_{r,3} > \times \mathcal{H} \langle : \phi_{0}^{2p} ; \Lambda_{r,3} >
$$

$$
\equiv \int_{(\mathbb{R}^{3})^{p}} \left\{ \int_{\mathbb{R}^{3}} \Lambda_{r,3}(\vec{y}) \prod_{k=1}^{2p} H_{-\frac{1}{m}}(\vec{y} - \vec{x}_{k}) \, dy \right\}
\left\{ \int_{\mathbb{R}^{3}} \Lambda_{r,3}(\vec{y}) \prod_{k=1}^{2p} H_{-\frac{1}{m}}(\vec{y} - \vec{x}_{k}) \, dy \right\}
\times : \hat{B}^{3}(\vec{x}_{1}) \cdots \hat{B}^{3}(\vec{x}_{2p}) \hat{B}^{3}(\vec{x}_{1}) \cdots \hat{B}^{3}(\vec{x}_{2p-1}) : \, d\vec{x}_{1} \cdots d\vec{x}_{2p-1}.
$$

(1.6)
Remark 2. i) To get a production between two \(<: \phi_0^{2p} : \Lambda_r, 3 >\), if we use the \(S\)-transform, then we may only have the first term of (1.6) and do not have the second term of (1.6).

ii) For the 6-th power \((<: \phi_0^{2p} : \Lambda_r, 3 >)^6\), the corresponding Hida product, denoted by \((<: \phi_0^{2p} : \Lambda_r, 3 >)^6\), involves much complicated terms. In particular, it possesses the following important term having a hexagonal form:

\[
\int_{\mathbb{R}^p} \int_{\mathbb{R}^p} \left\{ \Lambda_{r, 3}(y_1) \prod_{k=1}^{2p-2} H_{\mathcal{H}}(y_1 - x_{1, k}) \right\} \\
\times H_{\mathcal{H}}(y_2 - y_3) \left\{ \Lambda_{r, 3}(y_2) \prod_{k=1}^{2p-2} H_{\mathcal{H}}(y_2 - x_{2, k}) \right\} \\
\times H_{\mathcal{H}}(y_3 - y_1) \left\{ \Lambda_{r, 3}(y_3) \prod_{k=1}^{2p-2} H_{\mathcal{H}}(y_3 - x_{3, k}) \right\} \times \cdots \\
\cdots \times H_{\mathcal{H}}(y_6 - y_1) \left\{ \Lambda_{r, 3}(y_6) \prod_{k=1}^{2p-2} H_{\mathcal{H}}(y_6 - x_{6, k}) \right\} dy_1 \cdots dy_6
\]

\( : \tilde{B}^3(\tilde{x}_{1, 1}) \cdots \tilde{B}^3(\tilde{x}_{6, 2p-2}) : dx_{1, 1} \cdots dx_{6, 2p-2} \).

(1.7)

By using the Hida product we can define the power series of Hida distributions, in particular we can set

\[
(e^{-\lambda <: \phi_0^{2p} : \Lambda_r, 3 >})_{\mathcal{H}} \equiv \sum_{n=0}^{\infty} \frac{1}{n!} (\lambda <: \phi_0^{2p} : \Lambda_r, 3 >)^n_{\mathcal{H}}, \quad \lambda \in \mathbb{C}.
\]

(1.8)

Remark 3. By modifying the usual multiplication procedures of the random variables on the sharp time field to the Hida product, we can define the operator (cf. (1.14))

\[
(e^{it <: \phi_0^{2p} : \Lambda_r, 3 >})_{\mathcal{H}} e^{id\Gamma(H)} \phi_0(\varphi)e^{-it d\alpha(H)} \left( e^{-it <: \phi_0^{2p} : \Lambda_r, 3 >} \right)_{\mathcal{H}}
\]

with the domain \(\Gamma > 0\), which has the structure of the ring \(\mathcal{G}\).

[GrotS] defines the operators which are defined by making use of the \(S\)-transform, and [AGW] constructs the operators through the convolution of pseudo differential operators with generalized white noises, and they showed that these operators do not include the non trivial interactions. [NaMu] introduces an another important approach of the construction of the field by choosing special test functions that are not \(S\).

For \(d = 4\), every existing known result on the field operator with non trivial interaction, in particular \(\Phi_4^2\), is a statement that the field operator cannot be really an operator but a form on a Hilbert space and hence it does not admit an operation of productions.

□
References


[SiSi] Si Si: Poisson noise, infinite symmetric group and stochastic integrals based on $\dot{\mathcal{B}}(t)^2$: in *The Fifth Lévy Seminar*, Dec., 2006.