Supplement to "Family of the generalised gamma kernels: a generator of asymmetric kernels for nonnegative data"

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A Technical Proofs

A.1 Proof of Theorem 4

We concentrate on the case for interior x; the proof for boundary x is similar and thus omitted. We also employ a short-handed notation $K_i := K_{GG}(X_i; x, b)$ to save space. Then, it suffices to demonstrate that

$$Var\left\{\sqrt{nb^{1/2}}\hat{f}_{GG}(x)\right\} = Var\left(b^{1/4}K_i\right) + 2\sum_{\ell=1}^{n-1}\left(1-\frac{\ell}{n}\right)Cov\left(b^{1/4}K_i, b^{1/4}K_{i+\ell}\right) \sim V_I(2)\frac{f(x)}{\sqrt{x}}.$$

It follows from Theorem 1 that $Var\left(b^{1/4}K_i\right) \sim V_I(2) f(x)/\sqrt{x}$. Hence, we only need to show that

$$\sum_{\ell=1}^{n-1}\left(1-\frac{\ell}{n}\right)Cov\left(b^{1/4}K_i, b^{1/4}K_{i+\ell}\right) = o(1). \tag{A1}$$

Observe that the absolute value of the left-hand side of (A1) is bounded by

$$\sum_{\ell=1}^{\infty}\left|Cov\left(b^{1/4}K_i, b^{1/4}K_{i+\ell}\right)\right| = \left(\sum_{\ell=1}^{d_n} + \sum_{\ell=d_n+1}^{\infty}\right)\left|Cov\left(b^{1/4}K_i, b^{1/4}K_{i+\ell}\right)\right| = V_1 + V_2 \text{ (say)},$$

where the increasing sequence d_n is specified shortly. We evaluate V_2 first. By Davydov's lemma (e.g. Corollary A.2 of Hall and Heyde, 1980) and the stationarity of X_i,

$$\left|Cov\left(b^{1/4}K_i, b^{1/4}K_{i+\ell}\right)\right| \leq 8b^{1/2}\left(E\left|K_i - E\left(K_i\right)\right|^r\right)^{2/r}\alpha\left(\ell\right)^{1-2/r}. \tag{A2}$$

By C_r-inequality and $K_i \geq 0$, $E\left|K_i - E\left(K_i\right)\right|^r \leq 2^{r-1}\left[E\left(K_i^r\right) + \left\{E\left(K_i\right)\right\}^r\right]$. Because $E\left(K_i\right) = O\left(1\right)$ and $E\left(K_i^r\right) = O\left\{A_{b,r}\left(x\right)\right\} = O\left(b^{(1-r)/2}\right)$ by the proof of Theorem 1, we have

$$E\left|K_i - E\left(K_i\right)\right|^r = O\left(b^{\frac{1-r}{2}}\right). \tag{A3}$$

The size of the mixing coefficient also implies that

$$\alpha\left(\ell\right) \leq C_6\ell^{-q} \tag{A4}$$

for some constants $0 < C_6 < \infty$ and $q > \left(2 - 2/r\right)/\left(1 - 2/r\right)$. Substituting (A3) and (A4) into (A2) yields $\left|Cov\left(b^{1/4}K_i, b^{1/4}K_{i+\ell}\right)\right| \leq cb^{1/r-1/2}\ell^{-q(1-2/r)}$. Hence, $V_2 \leq cb^{1/r-1/2}\sum_{\ell=d_n+1}^{\infty}\ell^{-q(1-2/r)}$, where $q\left(1-2/r\right) > 1$ holds by construction. Also define $d_n := \lfloor b^{-a}\rfloor$ for some $a \in \left((1/2)\left\{q - 1/\left(1-2/r\right)\right\}^{-1}, 1/2\right)$. Then,

$$\sum_{\ell=d_n+1}^{\infty}\ell^{-q(1-2/r)} \leq \int_{d_n}^{\infty}x^{-q(1-2/r)}dx = \frac{d_n^{1-q(1-2/r)}}{q\left(1-2/r\right)-1} = O\left\{b^{a(q(1-2/r)-1)}\right\}, \tag{A5}$$

and thus $V_2 \leq O\left\{b^{a(q(1-2/r)-1)-(1/2)(1-2/r)}\right\} \to 0$.

We now turn to V_1. The stationarity of X_i and $K_i \geq 0$ imply that $\left|Cov\left(b^{1/4}K_i, b^{1/4}K_{i+\ell}\right)\right| \leq b^{1/2}\left[E\left(K_iK_{i+\ell}\right) + \left\{E\left(K_i\right)\right\}^2\right]$, where both $E\left(K_iK_{i+\ell}\right)$ and $E\left(K_i\right)$ are $O\left(1\right)$. Therefore, $V_1 \leq O\left(d_nb^{1/2}\right) = O\left(b^{1/2-a}\right) \to 0$, which establishes (A1). ■

A.2 Proof of Theorem 5

The proof requires four lemmata below. In particular, a Bernstein-type inequality for strong mixing processes in Lemma A4, which restates Theorem 2.1 of Liebscher (1996), constitutes the key part of the proof.

Lemma A1. *Let* $(\alpha_0, \beta_0, \gamma_0) := (\alpha_b(0), \beta_b(0), \gamma_b(0))$. *Then, for any* $\delta > 0$,

$$\int_0^\delta K_{GG}(u; 0, b)\, du = \int_0^\delta \frac{\gamma_0 u^{\alpha_0 - 1} \exp\left[-\left\{u / \left(\beta_0 \Gamma(\alpha_0/\gamma_0) / \Gamma((\alpha_0+1)/\gamma_0)\right)\right\}^{\gamma_0}\right]}{\left\{\beta_0 \Gamma(\alpha_0/\gamma_0) / \Gamma((\alpha_0+1)/\gamma_0)\right\}^{\alpha_0} \Gamma(\alpha_0/\gamma_0)} du \to 1,$$

as $b \to 0$.

Lemma A2. *For some* $\bar{x} \in [0, C_1 b)$, $K_{GG}(u; \bar{x}, b) \le C_7 b^{-1}$, *where*

$$C_7 := \left(\frac{C_4^2}{C_2}\right)(C_4+1)(C_4+2)\max\left\{1, (C_4-1)^{C_4-1}\right\}.$$

Lemma A3. *Let* $\bar{K}_i := K_{GG}(X_i; \bar{x}, b) - E\{K_{GG}(X_i; \bar{x}, b)\}$ *for* $\bar{x}$ *defined in Lemma A2. Then,* $E\left(\sum_{i=1}^m \bar{K}_i\right)^2 \le C_8 m b^{-2}$, *where*

$$C_8 := 2C_7^2 \left[1 + 32 C_6^{1-2/r}\left\{1 + \frac{1}{q(1-2/r) - 1}\right\}\right].$$

Lemma A4. (Liebscher, 1996, Theorem 2.1) *Let* $\{Z_i\}$ *be a strictly stationary and strong mixing process with the mixing coefficient* $\alpha(\ell)$ *such that* $E(Z_i) = 0$ *and* $|Z_i| \le S(n), i = 1, \ldots, n$. *Then, for any integer* $1 \le m \le n$ *and for any* $\epsilon > 4mS(n)$,

$$\Pr\left(\left|\sum_{i=1}^n Z_i\right| > \epsilon\right) \le 4\exp\left\{-\frac{\epsilon^2}{64(n/m)\sigma^2(m) + (8/3)\epsilon m S(n)}\right\} + 4\frac{n}{m}\alpha(m),$$

where $\sigma^2(m) := E\left(\sum_{i=1}^m Z_i\right)^2$.

A.2.1 Proof of Lemma A1

By the change of variable $v := [u/\{\beta_0 \Gamma(\alpha_0/\gamma_0)/\Gamma((\alpha_0+1)/\gamma_0)\}]^{\gamma_0}$, the integral can be rewritten as $\int_0^{C_\delta} \left\{v^{(\alpha_0/\gamma_0)-1} \exp(-v)/\Gamma(\alpha_0/\gamma_0)\right\} dv$, where the integrand is the pdf of $G(\alpha_0/\gamma_0, 1)$, and

$$C_\delta = \left[\frac{\delta}{\beta_0 \Gamma(\alpha_0/\gamma_0)/\Gamma\{(\alpha_0+1)/\gamma_0\}}\right]^{\gamma_0}.$$

Therefore, the proof is boiled down to showing that for any $\delta > 0$, $C_\delta \to \infty$ as $b \to 0$.

Recognizing $\alpha_0 \in [1, C_4]$ and $\gamma_0 \geq 1$, we deduce that $\alpha_0 = O(\gamma_0)$ or $\alpha_0 = o(\gamma_0)$ must be the case. If $\alpha_0 = O(\gamma_0)$, then $\Gamma(\alpha_0/\gamma_0)$ and $\Gamma\{(\alpha_0+1)/\gamma_0\}$ are both $O(1)$. It follows from Condition 2 that $C_\delta = O\left(\beta_0^{-\gamma_0}\right) = O(b^{-\gamma_0}) \to \infty$. Alternatively, if $\alpha_0 = o(\gamma_0)$, then we may pick an arbitrarily small b so that $|\alpha_0/\gamma_0| \leq 1$ and $|(\alpha_0+1)/\gamma_0| \leq 1$. Using SELG and the property of the gamma function yields

$$\log\Gamma(z) = -\log(z) - \gamma z + \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k} z^k$$

for $z = \alpha_0/\gamma_0, (\alpha_0+1)/\gamma_0$. Then,

$$\begin{aligned} \frac{\Gamma(\alpha_0/\gamma_0)}{\Gamma\{(\alpha_0+1)/\gamma_0\}} &= \exp\left\{\log\Gamma\left(\frac{\alpha_0}{\gamma_0}\right) - \log\Gamma\left(\frac{\alpha_0+1}{\gamma_0}\right)\right\} \\ &= \left(1+\frac{1}{\alpha_0}\right)\exp\left[O\left(\frac{1}{\gamma_0}\right) + O\left\{\left(\frac{\alpha_0}{\gamma_0}\right)^2\right\}\right] = O(1), \end{aligned}$$

and thus it again holds that $C_\delta = O(b^{-\gamma_0}) \to \infty$. ■

A.2.2 Proof of Lemma A2

Let $\left(\bar{\alpha}, \bar{\beta}, \bar{\gamma}\right) := (\alpha_b(\bar{x}), \beta_b(\bar{x}), \gamma_b(\bar{x}))$. The upper bound can be implied by $K_{GG}(u^*; \bar{x}, b)$, where u^* is the mode. Because the shape of $K_{GG}(u; \bar{x}, b)$ is substantially different between the cases with $\bar{\alpha} = 1$ and $\bar{\alpha} > 1$, we evaluate two cases separately.

When $\bar{\alpha} > 1$, a straightforward calculation yields

$$u^* = \left[\frac{\bar{\beta}\Gamma(\bar{\alpha}/\bar{\gamma})}{\Gamma\{(\bar{\alpha}+1)/\bar{\gamma}\}}\right]\left(\frac{\bar{\alpha}-1}{\bar{\gamma}}\right)^{1/\bar{\gamma}}$$

so that

$$K_{GG}(u^*; \bar{x}, b) = \left(\frac{\bar{\gamma}}{\bar{\beta}}\right)\left[\frac{\Gamma\{(\bar{\alpha}+1)/\bar{\gamma}\}}{\Gamma^2(\bar{\alpha}/\bar{\gamma})}\right]\left(\frac{\bar{\alpha}-1}{\bar{\gamma}}\right)^{(\bar{\alpha}-1)/\bar{\gamma}} \exp\left\{-\left(\frac{\bar{\alpha}-1}{\bar{\gamma}}\right)\right\}. \quad \text{(A6)}$$

Observe that $\Gamma\{(\bar{\alpha}+1)/\bar{\gamma}\}/\Gamma^2(\bar{\alpha}/\bar{\gamma}) = \{\Gamma(2\bar{\alpha}/\bar{\gamma})/\Gamma^2(\bar{\alpha}/\bar{\gamma})\}[\Gamma\{(\bar{\alpha}+1)/\bar{\gamma}\}/\Gamma(2\bar{\alpha}/\bar{\gamma})]$. It follows from Corollary 1 of Cerone (2007) and the property of the gamma function that for $z > 0$,

$$\frac{\Gamma^2(1+z)}{\Gamma(2+2z)} \geq \frac{1}{(1+z)^{2+z}} \Rightarrow \frac{\Gamma(2z)}{\Gamma^2(z)} \leq \frac{z(1+z)^{2+z}}{2(1+2z)}. \tag{A7}$$

Putting $z = \bar{\alpha}/\bar{\gamma}$ gives

$$\frac{\Gamma(2\bar{\alpha}/\bar{\gamma})}{\Gamma^2(\bar{\alpha}/\bar{\gamma})} \leq \left(\frac{1}{2}\right)\left(\frac{\bar{\alpha}}{\bar{\gamma}+2\bar{\alpha}}\right)\left(1+\frac{\bar{\alpha}}{\bar{\gamma}}\right)^{2+\bar{\alpha}/\bar{\gamma}}. \tag{A8}$$

Moreover, by Theorem 1 of Kečkić and Vasić (1971) and the property of the gamma function, for $x > y > 0$,

$$\frac{\Gamma(1+x)}{\Gamma(1+y)} \geq \frac{(1+x)^x}{(1+y)^y}\exp(y-x) \Rightarrow \frac{\Gamma(y)}{\Gamma(x)} \leq \frac{x(1+y)^y}{y(1+x)^x}\exp(x-y).$$

Letting $(x,y) = (2\bar{\alpha}/\bar{\gamma}, (\bar{\alpha}+1)/\bar{\gamma})$, we have

$$\frac{\Gamma\{(\bar{\alpha}+1)/\bar{\gamma}\}}{\Gamma(2\bar{\alpha}/\bar{\gamma})} \leq \left(\frac{2\bar{\alpha}}{1+\bar{\alpha}}\right)\frac{\{1+(\bar{\alpha}+1)/\bar{\gamma}\}^{(\bar{\alpha}+1)/\bar{\gamma}}}{(1+2\bar{\alpha}/\bar{\gamma})^{2\bar{\alpha}/\bar{\gamma}}}\exp\left(\frac{\bar{\alpha}-1}{\bar{\gamma}}\right). \tag{A9}$$

Substituting (A8) and (A9) into (A6), rearranging it, and then using $\bar{\alpha} \in (1, C_4]$, $\bar{\beta} \geq C_2 b$ and $\bar{\gamma} \geq 1$, we deduce that

$$\begin{aligned} K_{GG}(u^*;\bar{x},b) &\leq \left(\frac{1}{\bar{\beta}}\right)\left(\frac{\bar{\gamma}}{\bar{\gamma}+2\bar{\alpha}}\right)\left(\frac{\bar{\alpha}^2}{\bar{\alpha}+1}\right)\left[\frac{(1+\bar{\alpha}/\bar{\gamma})\{1+(\bar{\alpha}+1)/\bar{\gamma}\}}{(1+2\bar{\alpha}/\bar{\gamma})^2}\right]^{\bar{\alpha}/\bar{\gamma}} \\ &\quad \cdot \left(\frac{\bar{\alpha}}{\bar{\gamma}}+1\right)^2\left(\frac{\bar{\alpha}+1}{\bar{\gamma}}+1\right)^{1/\bar{\gamma}}\left(\frac{\bar{\alpha}-1}{\bar{\gamma}}\right)^{(\bar{\alpha}-1)/\bar{\gamma}} \\ &\leq \left(\frac{1}{C_2 b}\right)\cdot 1\cdot\left(\frac{C_4^2}{C_4+1}\right)\cdot 1\cdot(C_4+1)^2\cdot(C_4+2)\cdot\max\left\{1,(C_4-1)^{C_4-1}\right\}. \end{aligned}$$

In sum, as far as $\bar{\alpha} > 1$, $K_{GG}(u;\bar{x},b) \leq C_7 b^{-1}$, where

$$C_7 := \left(\frac{C_4^2}{C_2}\right)(C_4+1)(C_4+2)\max\left\{1,(C_4-1)^{C_4-1}\right\}.$$

On the other hand, when $\bar{\alpha} = 1$, it follows from (A7) and $u^* = 0$ that

$$\begin{aligned} K_{GG}(u^*;\bar{x},b) &= \left(\frac{\bar{\gamma}}{\bar{\beta}}\right)\left\{\frac{\Gamma(2/\bar{\gamma})}{\Gamma^2(1/\bar{\gamma})}\right\} \\ &\leq \left(\frac{1}{\bar{\beta}}\right)\left\{\frac{\bar{\gamma}}{2(\bar{\gamma}+2)}\right\}\left(1+\frac{1}{\bar{\gamma}}\right)^{2+1/\bar{\gamma}} \\ &\leq \left(\frac{1}{C_2 b}\right)\cdot\left(\frac{1}{2}\right)\cdot 2^3 = \left(\frac{4}{C_2}\right)b^{-1}. \end{aligned}$$

Note that $C_7 \geq 6/C_2$ holds, which establishes the lemma. ■

A.2.3 Proof of Lemma A3

By the stationarity of Z_i,

$$E\left(\sum_{i=1}^{m}\bar{K}_i\right)^2 \leq mE\left(\bar{K}_i^2\right)+2\sum_{\ell=1}^{m-1}(m-\ell)\,E\left|\bar{K}_i\bar{K}_{i+\ell}\right|, \tag{A10}$$

where, by Lemma A2 and $\int_0^\infty f(u)\,du=1$,

$$E\left(\bar{K}_i^2\right)\leq E\left|K_{GG}(X_i;\bar{x},b)\right|^2+E^2\left|K_{GG}(X_i;\bar{x},b)\right|\leq 2C_7^2b^{-2}. \tag{A11}$$

Following the same manner as in the proof of Theorem 4, we also have $E\left|\bar{K}_i\bar{K}_{i+\ell}\right|\leq 8\left(E\left|\bar{K}_i\right|^r\right)^{2/r}\alpha(\ell)^{1-2/r}$, where, by C_r-inequality, Lemma A2 and $\int_0^\infty f(u)\,du=1$,

$$E\left|\bar{K}_i\right|^r\leq 2^{r-1}\left\{E\left|K_{GG}(X_i;\bar{x},b)\right|^r+E^r\left|K_{GG}(X_i;\bar{x},b)\right|\right\}\leq\left(2C_7b^{-1}\right)^r.$$

Therefore, $E\left|\bar{K}_i\bar{K}_{i+\ell}\right|\leq 32C_6^{1-2/r}C_7^2b^{-2}\ell^{-q(1-2/r)}$ by (A4), and thus

$$\begin{aligned}\sum_{\ell=1}^{m-1}(m-\ell)\,E\left|\bar{K}_i\bar{K}_{i+\ell}\right| &\leq 32C_6^{1-2/r}C_7^2mb^{-2}\sum_{\ell=1}^{\infty}\ell^{-q(1-2/r)}\\ &\leq 32C_6^{1-2/r}C_7^2\left\{1+\frac{1}{q(1-2/r)-1}\right\}mb^{-2}, \end{aligned}\tag{A12}$$

where the last inequality follows from (A5). Combining (A10), (A11) and (A12) establishes the result. ■

A.2.4 Proof of Theorem 5

This proof largely follows the one of Proposition 3.3 in Bouezmarni and Van Bellegem (2011). The proof completes if the following statements hold for some $\bar{x}\in[0,C_1b)$:

$$\hat{f}_{GG}(\bar{x}) = E\left\{\hat{f}_{GG}(\bar{x})\right\}+o_p(1). \tag{A13}$$

$$E\left\{\hat{f}_{GG}(\bar{x})\right\} = E\left\{\hat{f}_{GG}(0)\right\}+o(1). \tag{A14}$$

$$E\left\{\hat{f}_{GG}(0)\right\} \to \infty. \tag{A15}$$

Note that (A14) immediately follows from the fact that $K_{GG}(u;\bar{x},b)\to K_{GG}(u;0,b)$ as $\bar{x}\to 0$.

We demonstrate (A15) first. When $f(x) \to \infty$ as $x \to 0$, it holds that for any $A > 0$, there is some $\delta > 0$ such that $f(x) > A$ for all $x < \delta$. For the given δ, Lemma A1 implies that

$$E\left\{\hat{f}_{GG}(0)\right\} > \int_0^{\delta} K_{GG}(u;0,b)\, f(u)\, du > A \int_0^{\delta} K_{GG}(u;0,b)\, du \to A,$$

which establishes (A15).

To show (A13), consider $\bar{K}_i$ in Lemma A3. Then, $E\left(\bar{K}_i\right) = 0$ and the same logic as applied for (A11) establishes that $\left|\bar{K}_i\right| \le 2C_7 b^{-1}$. Also pick $b = O\left(n^{-\eta}\right)$ for some $\eta \in (0, 1/2)$ and $m = \lfloor n^a \rfloor$ for some $a \in (\max\{\eta, 1/(1+q)\}, 1/2)$ for concreteness. Then, for a sufficiently large n, $1 \le m \le n$ holds. Because $m(nb)^{-1} = O\left\{n^{a-(1-\eta)}\right\} \to 0$, we also have $n\epsilon > 8C_7 m b^{-1}$ for an arbitrarily chosen $\epsilon > 0$. Therefore, for the given ϵ, we may apply Lemmata A3 and A4 and (A4) to obtain

$$\begin{aligned}
& \Pr\left(\left|\hat{f}_{GG}(\bar{x}) - E\left\{\hat{f}_{GG}(\bar{x})\right\}\right| > \epsilon\right) \\
= \; & \Pr\left(\left|\sum_{i=1}^{n} \bar{K}_i\right| > n\epsilon\right) \\
\le \; & 4\exp\left\{-\frac{(n\epsilon)^2}{64(n/m)\left(C_8 m b^{-2}\right) + (8/3)(n\epsilon)\, m\left(2C_7 b^{-1}\right)}\right\} + 4\frac{n}{m}\left(C_6 m^{-q}\right) \\
= \; & 4\exp\left\{-\frac{3\epsilon^2\left(nb^2\right)}{16\left(12C_8 + C_7\epsilon m b\right)}\right\} + 4C_6 n m^{-(1+q)}. \qquad \text{(A16)}
\end{aligned}$$

Since $mb = O\left(n^{a-\eta}\right) \to \infty$, a geometric series expansion to (the absolute value of) the exponent of the first term yields

$$\begin{aligned}
\frac{3\epsilon^2\left(nb^2\right)}{16\left(12C_8 + C_7\epsilon m b\right)} &= \left(\frac{3\epsilon}{16C_7}\right)\left[\frac{1}{1 + \left\{12C_8/\left(C_7\epsilon\right)\right\}(mb)^{-1}}\right]\left(\frac{nb}{m}\right) \\
&= \left(\frac{3\epsilon}{16C_7}\right)\left[1 - \left(\frac{12C_8}{C_7\epsilon}\right)(mb)^{-1} + O\left\{(mb)^{-2}\right\}\right]\left(\frac{nb}{m}\right).
\end{aligned}$$

Therefore, the right-hand side of (A16) is bounded by $O\left\{\exp\left(-(3\epsilon)(16C_7)^{-1}(nb/m)\right)\right\} + O\left\{n^{-(a(1+q)-1)}\right\} \to 0$, which completes the proof. ∎

A.3 Proof of Theorem 6

This proof largely follows the one of Theorem 5.3 in Bouezmarni and Scaillet (2005). For some $\bar{x} \in [0, C_1 b)$, the proof is boiled down to establishing the following statements:

$$\left| \frac{E\left\{\hat{f}_{GG}(\bar{x})\right\} - f(\bar{x})}{f(\bar{x})} \right| \to 0. \tag{A17}$$

$$\left| \frac{\hat{f}_{GG}(\bar{x}) - E\left\{\hat{f}_{GG}(\bar{x})\right\}}{f(\bar{x})} \right| \xrightarrow{p} 0. \tag{A18}$$

We demonstrate (A17) first. Although Theorem 5.3 of Bouezmarni and Scaillet (2005) is based on random sampling, their proof strategy still works for (A17). An inspection of the proof reveals that (A17) is shown if their conditions A.2, A.3 and A.5 are fulfilled. Because $\int_0^\infty f(x)\,dx = 1$ and $f(x) \to \infty$ as $x \to 0$, there are constants $0 < C_9, C_{10} < \infty$ such that $C_9 x^{-d} \le f(x) \le C_{10} x^{-d}$ for some $d \in (0,1)$ as $x \to 0$. Accordingly, $f'(x) = O\left(x^{-d-1}\right)$ for a small value of x. These imply that $x\,|f'(x)|\,/f(x) \le O(1)$, and thus A.2 follows. Next, a minor modification of the proof of Lemma A1 establishes that for any $\delta > 0$, $\int_0^\delta K_{GG}(u; \bar{x}, b)\,du \to 1$ as $\bar{x}, b \to 0$; indeed, the argument in the proof still holds after replacing $(\alpha_0, \beta_0, \gamma_0)$ with $\left(\bar{\alpha}, \bar{\beta}, \bar{\gamma}\right) = \left(\alpha_b(\bar{x}), \beta_b(\bar{x}), \gamma_b(\bar{x})\right)$. Hence, A.3 is also valid. Finally, the proof of Theorem 1 and Conditions 1 and 3 suggest that $Var(\theta_{\bar{x}}) = \bar{\beta}^2 \{M_b(\bar{x}) - 1\} = O(b^2)$ as $\bar{x}, b \to 0$ for $\theta_{\bar{x}} \stackrel{d}{=} GG\left(\bar{\alpha}, \bar{\beta}\Gamma(\bar{\alpha}/\bar{\gamma})/\Gamma\{(\bar{\alpha}+1)/\bar{\gamma}\}, \bar{\gamma}\right)$. Therefore, A.5 is also established, and thus (A17) is proven.

To show (A18) under dependent sampling, we rely on Lemma A4 as in the proof of Theorem 5 above. For $\bar{K}_i$ defined in Lemma A3, $E\left(\bar{K}_i\right) = 0$ and $\left|\bar{K}_i\right| \le 2C_7 b^{-1}$. We again pick $b = O(n^{-\eta})$ for some $\eta \in (0, 1/2)$ and $m = \lfloor n^a \rfloor$ for some $a \in (\max\{\eta, 1/(1+q)\}, 1/2)$. Then, for a sufficiently large n, $1 \le m \le n$ holds. Because

$m\left\{nbf\left(\bar{x}\right)\right\}^{-1}=O\left\{n^{a-(1-\eta)}f^{-1}\left(\bar{x}\right)\right\}\to 0$ as $\bar{x}\to 0$, we also have $nf\left(\bar{x}\right)\epsilon>8C_7mb^{-1}$ for an arbitrarily chosen $\epsilon>0$. Therefore, for the given ϵ, we may apply Lemmata A3-A4 and (A4) to obtain

$$\begin{aligned}
&\Pr\left(\left|\frac{\hat{f}_{GG}\left(\bar{x}\right)-E\left\{\hat{f}_{GG}\left(\bar{x}\right)\right\}}{f\left(\bar{x}\right)}\right|>\epsilon\right)\\
=\;&\Pr\left(\left|\sum_{i=1}^{n}\bar{K}_i\right|>nf\left(\bar{x}\right)\epsilon\right)\\
\leq\;&4\exp\left[-\frac{\left\{nf\left(\bar{x}\right)\epsilon\right\}^2}{64\left(n/m\right)\left(C_8mb^{-2}\right)+\left(8/3\right)\left\{nf\left(\bar{x}\right)\epsilon\right\}m\left(2C_7b^{-1}\right)}\right]+4\frac{n}{m}\left(C_6m^{-q}\right)\\
=\;&4\exp\left[-\left(\frac{3\epsilon}{16C_7}\right)\left\{1-\left(\frac{12C_8}{C_7\epsilon}\right)\left(mbf\left(\bar{x}\right)\right)^{-1}+O\left(\left(mbf\left(\bar{x}\right)\right)^{-2}\right)\right\}\left\{\frac{nbf\left(\bar{x}\right)}{m}\right\}\right]\\
&+4C_6nm^{-(1+q)},
\end{aligned}\qquad\text{(A19)}$$

where the geometric series expansion in the final equality comes from the fact that $mbf\left(\bar{x}\right)=O\left\{n^{a-\eta}f\left(\bar{x}\right)\right\}\to\infty$. Therefore, the right-hand side of (A19) is bounded by $O\left\{\exp\left(-\left(3\epsilon\right)\left(16C_7\right)^{-1}\left(nbf\left(\bar{x}\right)/m\right)\right)\right\}+O\left\{n^{-(a(1+q)-1)}\right\}\to 0$, which completes the proof. ■

B Comprehensive Simulation Results

Table B1 below presents expanded simulation results. In addition to six density estimators reported in Section 4, the density estimator using the Gaussian kernel is included as a symmetric kernel density estimator ("S") in the original scale. Besides, while the tuning parameter (i.e. b or h) for each estimator mentioned so far is chosen as the minimizer of the (approximated) RISE, the rule-of-thumb smoothing parameter in Section 2.2.3 is also examined for W, NM and MG. Asterisks indicate the estimators with this type of smoothing parameter plugged in.

Table B1: Averages of Performance Measures and Tuning Parameter Values

		n = 100		n = 200		n = 500	
		RISE	b or h	RISE	b or h	RISE	b or h
				1. Gamma			
GG	W	0.0356 (0.0098)	0.2778	0.0294 (0.0081)	0.1701	0.0221 (0.0057)	0.0897
	NM	0.0368 (0.0091)	0.3007	0.0306 (0.0076)	0.1861	0.0232 (0.0055)	0.0975
	MG	0.0362 (0.0112)	0.1712	0.0289 (0.0088)	0.1105	0.0211 (0.0059)	0.0683
	W*	0.0392 (0.0115)	0.1653	0.0311 (0.0088)	0.1261	0.0229 (0.0058)	0.0873
	NM*	0.0401 (0.0099)	0.2584	0.0330 (0.0077)	0.1971	0.0252 (0.0052)	0.1365
	MG*	0.0385 (0.0120)	0.1292	0.0302 (0.0091)	0.0985	0.0219 (0.0060)	0.0682
Non-GG	G	0.0358 (0.0125)	0.1404	0.0290 (0.0098)	0.0962	0.0220 (0.0066)	0.0601
	S	0.0415 (0.0123)	0.2937	0.0337 (0.0090)	0.2401	0.0256 (0.0062)	0.1831
	LT	0.0441 (0.0157)	0.4434	0.0348 (0.0114)	0.3820	0.0252 (0.0074)	0.3149
	LL	0.0368 (0.0116)	1.0272	0.0302 (0.0088)	0.7524	0.0234 (0.0061)	0.5152
				2. Weibull			
GG	W	0.0374 (0.0119)	0.1870	0.0297 (0.0090)	0.1228	0.0214 (0.0058)	0.0809
	NM	0.0385 (0.0116)	0.2090	0.0307 (0.0088)	0.1382	0.0222 (0.0058)	0.0911
	MG	0.0367 (0.0127)	0.1272	0.0286 (0.0092)	0.0915	0.0204 (0.0060)	0.0634
	W*	0.0414 (0.0132)	0.1102	0.0320 (0.0093)	0.0840	0.0228 (0.0060)	0.0585
	NM*	0.0414 (0.0127)	0.1721	0.0323 (0.0091)	0.1313	0.0231 (0.0059)	0.0914
	MG*	0.0400 (0.0132)	0.0861	0.0308 (0.0094)	0.0657	0.0218 (0.0061)	0.0457
Non-GG	G	0.0368 (0.0140)	0.1137	0.0294 (0.0103)	0.0813	0.0218 (0.0069)	0.0526
	S	0.0375 (0.0123)	0.3167	0.0304 (0.0090)	0.2647	0.0227 (0.0059)	0.2083
	LT	0.0470 (0.0154)	0.3730	0.0368 (0.0110)	0.3187	0.0267 (0.0073)	0.2584
	LL	0.0367 (0.0127)	0.8234	0.0294 (0.0092)	0.6600	0.0217 (0.0060)	0.5098
				3. Half-Normal			
GG	W	0.0274 (0.0113)	0.3445	0.0225 (0.0083)	0.2745	0.0172 (0.0056)	0.1936
	NM	0.0251 (0.0120)	0.3809	0.0207 (0.0088)	0.3152	0.0158 (0.0059)	0.2354
	MG	0.0303 (0.0117)	0.2662	0.0245 (0.0087)	0.2039	0.0184 (0.0060)	0.1380
	W*	0.0413 (0.0148)	0.1164	0.0323 (0.0107)	0.0886	0.0234 (0.0069)	0.0616
	NM*	0.0361 (0.0141)	0.1819	0.0288 (0.0104)	0.1385	0.0211 (0.0068)	0.0962
	MG*	0.0423 (0.0154)	0.0910	0.0328 (0.0111)	0.0692	0.0235 (0.0071)	0.0481
Non-GG	G	0.0327 (0.0112)	0.1750	0.0262 (0.0087)	0.1325	0.0193 (0.0057)	0.0911
	S	0.0531 (0.0122)	0.2579	0.0461 (0.0092)	0.1946	0.0375 (0.0067)	0.1276
	LT	0.0586 (0.0191)	0.4735	0.0457 (0.0140)	0.3997	0.0329 (0.0085)	0.3224
	LL	0.0256 (0.0119)	1.4774	0.0203 (0.0094)	1.2598	0.0147 (0.0064)	1.0193

Table B1: *Continued*

		n = 100		n = 200		n = 500	
		RISE	b or h	RISE	b or h	RISE	b or h
				4. Log-Normal			
GG	W	0.0429 (0.0153)	0.0830	0.0332 (0.0110)	0.0654	0.0242 (0.0080)	0.0471
	NM	0.0447 (0.0152)	0.0932	0.0343 (0.0108)	0.0749	0.0245 (0.0078)	0.0562
	MG	0.0416 (0.0158)	0.0624	0.0324 (0.0114)	0.0480	0.0238 (0.0081)	0.0334
	W*	0.0543 (0.0159)	0.1584	0.0416 (0.0118)	0.1213	0.0292 (0.0087)	0.0852
	NM*	0.0715 (0.0153)	0.2476	0.0564 (0.0123)	0.1896	0.0387 (0.0091)	0.1331
	MG*	0.0531 (0.0170)	0.1238	0.0413 (0.0127)	0.0948	0.0299 (0.0091)	0.0666
Non-GG	G	0.0458 (0.0150)	0.0535	0.0360 (0.0108)	0.0390	0.0263 (0.0075)	0.0261
	S	0.0527 (0.0133)	0.1929	0.0422 (0.0095)	0.1565	0.0313 (0.0067)	0.1205
	LT	0.0401 (0.0166)	0.3207	0.0315 (0.0122)	0.2782	0.0232 (0.0082)	0.2310
	LL	0.0482 (0.0147)	0.4415	0.0381 (0.0106)	0.3683	0.0282 (0.0073)	0.2937
				5. Generalized Champernowne			
GG	W	0.0477 (0.0169)	0.1324	0.0391 (0.0126)	0.0922	0.0295 (0.0101)	0.0547
	NM	0.0477 (0.0161)	0.1448	0.0390 (0.0122)	0.1033	0.0294 (0.0099)	0.0637
	MG	0.0504 (0.0190)	0.0881	0.0403 (0.0141)	0.0618	0.0298 (0.0106)	0.0388
	W*	0.0571 (0.0252)	0.1796	0.0458 (0.0195)	0.1397	0.0358 (0.0174)	0.1052
	NM*	0.0652 (0.0327)	0.2807	0.0525 (0.0251)	0.2184	0.0420 (0.0230)	0.1645
	MG*	0.0598 (0.0254)	0.1404	0.0482 (0.0198)	0.1092	0.0372 (0.0176)	0.0822
Non-GG	G	0.0504 (0.0195)	0.0676	0.0403 (0.0151)	0.0485	0.0301 (0.0107)	0.0318
	S	0.0623 (0.0195)	0.1333	0.0513 (0.0139)	0.1100	0.0404 (0.0094)	0.0845
	LT	0.0700 (0.0246)	0.4451	0.0544 (0.0177)	0.3807	0.0394 (0.0117)	0.3118
	LL	0.0513 (0.0183)	0.4850	0.0413 (0.0137)	0.3618	0.0308 (0.0101)	0.2637
				6. Gamma with Pole			
GG	W	0.0617 (0.0181)	0.0858	0.0494 (0.0136)	0.0591	0.0359 (0.0091)	0.0380
	NM	0.0652 (0.0168)	0.0855	0.0523 (0.0128)	0.0586	0.0380 (0.0088)	0.0373
	MG	0.0627 (0.0171)	0.0803	0.0500 (0.0131)	0.0554	0.0368 (0.0098)	0.0357
	W*	0.0773 (0.0172)	0.1772	0.0644 (0.0128)	0.1365	0.0497 (0.0084)	0.0947
	NM*	0.0974 (0.0150)	0.2770	0.0830 (0.0110)	0.2133	0.0660 (0.0073)	0.1479
	MG*	0.0739 (0.0153)	0.1385	0.0607 (0.0111)	0.1066	0.0465 (0.0075)	0.0740
Non-GG	G	0.0614 (0.0202)	0.0571	0.0494 (0.0157)	0.0389	0.0363 (0.0107)	0.0242
	S	0.1116 (0.0178)	0.0779	0.0954 (0.0137)	0.0523	0.0751 (0.0092)	0.0316
	LT	0.0639 (0.0304)	0.7389	0.0498 (0.0213)	0.6298	0.0361 (0.0140)	0.5138
	LL	0.0650 (0.0161)	0.5280	0.0549 (0.0125)	0.3905	0.0438 (0.0099)	0.2552

Note: Numbers in parentheses are simulation standard deviations of RISEs. "b or h" denotes simulation averages of the values of smoothing parameters b for W, NM, MG, and G, or the lengths of bandwidths h for S, LT and LL. Estimators with asterisks are those with rule-of-thumb smoothing parameters plugged in.

References

Bouezmarni, T., and Scaillet, O. (2005), 'Consistency of Asymmetric Kernel Density Estimators and Smoothed Histograms with Application to Income Data', *Econometric Theory*, 21, 390-412.

Bouezmarni, T., and Van Bellegem, S. (2011), 'Nonparametric Beta Kernel Estimator for Long Memory Time Series', ECORE Discussion Paper 2011/19.

Cerone, P. (2007), 'Special Functions: Approximations and Bounds', *Applicable Analysis and Discrete Mathematics*, 1, 72-91.

Hall, P., and Heyde, C. C. (1980), *Martingale Limit Theory and Its Application*, New York: Academic Press.

Kečkić, J. D., and Vasić, P. M. (1971), 'Some Inequalities for the Gamma Function', *Publications de l'Institut Mathématique*, 11, 107-114.

Liebscher, E. (1996), 'Strong Convergence of Sums of α-Mixing Random Variables with Applications to Density Estimation', *Stochastic Processes and Their Applications*, 65, 69-80.